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Mesonic correlation lengths in high-temperature QCD

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Abstract: We consider spatial correlation lengths ξ for various QCD light quark bilinears, at temperatures above a few hundred MeV. Some of the correlation lengths (such as that related to baryon density) coincide with what has been measured earlier on from glueball-like states; others do not couple to glueballs, and have a well-known perturbative leading-order expression as well as a computable next-to-leading-order correction. We determine the latter following analogies with the NRQCD effective theory, used for the study of heavy quarkonia at zero temperature: we find (for the quenched case) $\xi^{-1} = 2\pi T + 0.1408 g^2 T$, and compare with lattice results. One manifestation of $\mathrm{U_A}(1)$ symmetry non-restoration is also pointed out.

Keywords: Thermal Field Theory, Physics of the Early Universe, NLO Computations, Lattice QCD.

Contents

1. Introduction

Given the study of quark-gluon plasma at RHIC and LHC, there is a continuing need for understanding what QCD actually predicts for the various physical observables of interest, in case the system does reach thermodynamical equilibrium. This is a highly non-trivial task, due to the strong interactions present in QCD even at temperatures above a few hundred MeV. Some of the observables can in principle be determined from first principles using lattice methods, but at present systematic errors, related particularly to light dynamical quarks, might be substantial. Therefore, there is a demand for alternative tools, whose results could be compared with those from lattice simulations.

Because of asymptotic freedom, a natural candidate for such a tool is perturbation theory. Alas, it suffers from serious infrared (IR) problems at high temperatures [1, 2]. These problems are of a rather specific kind, however: they concern directly only static, long-range gluons. This has lead to the general framework of dimensionally reduced effective field theories [3]. The idea is that there are still a number of computations which can be carried out reliably in perturbation theory, while the parts which cannot, can be treated with lattice simulations using an effective theory which is much simpler than the original one, all the quarks having been integrated out.

During the last few years, the program of dimensional reduction has been applied to a large variety of physical quantities: in particular, to various gluonic correlation lengths [4]–[10] as well as the equation of state [11]–[14]. Comparisons with 4d lattice results in pure SU(2) and SU(3) Yang-Mills theories are summarised in [7, 9, 15, 16, 17], and they suggest that dimensional reduction is applicable as soon as the temperature is somewhat above T_c, where T_c is the critical temperature for the deconfinement phase transition.

An even simpler pattern can perhaps be argued for [12, 15, 17]: it appears, inspecting the empirical results from the studies mentioned, that a truncated coupling constant series may already give a reasonable part of the full answer, if carried out to a sufficient depth such that all dynamical scales in the system have made their entrances. Therefore, either non-perturbative studies with the dimensionally reduced theory, or the further reduction suggested by this prescription, may provide a welcome alternative to direct 4d lattice simulations of QCD, particularly for observables where the systematic errors are not well under control. It should be mentioned that other proposals for analytic tools exist, as well (for recent reviews, see [18]).

An important class of observables which have attracted somewhat less attention in these settings, are those built out of quark fields, rather than gluonic fields. In general, they are expected to be less IR sensitive and thus more perturbative than purely gluonic observables, so that analytic tools should be better applicable. At the same time, there is more urgency for analytic tools, given the severe difficulties in treating light dynamical fermions on the lattice. On top of these formal motivations, "mesonic" and "baryonic" observables may be more directly related to experimental signatures than gluonic ones. During the last few years, significant progress has been made on one class of observables of this type, that is various quark number susceptibilities (see, e.g., [19]–[24]). The purpose of this paper is to address another class, closely related but more directly sensitive to IR physics: the long-distance spatial correlation lengths related to mesonic operators (i.e., gauge-invariant quark bilinears) constructed out of the light quark flavours [25].

There is a large variety of independent mesonic observables of the type mentioned. Assuming, as we will in this paper, QCD to consist of N_f flavours of degenerate quarks, they can be divided, first of all, into singlets and non-singlets under the flavour group. Each class can furthermore be divided into scalar, pseudoscalar, vector, and axial vector objects. Physically perhaps the most interesting quark bilinears are those representing baryon number density and electric charge density, but since all the other ones can in principle be measured on the lattice equally well and this is the most immediate reference point for our results, we will try to be as general as possible.

Previous analytic work on similar topics was initiated more than a decade ago [26]–[28], following the first numerical studies [25]. (There was even earlier work for small temperatures, $0 < T \ll T_c$ [29, 30].) Many of these studies had mostly qualitative motivations, however: corrections of order unity with respect to the ones computed were (knowingly) left out. They were also partly motivated by some qualitative features in the early lattice results (particularly that the vector channel, or ρ, appeared to be significantly "heavier" than the pseudoscalar channel, or π, even at $T > T_c$), which have turned out to be finite lattice spacing artifacts. Significant further progress was made in [31], and we show here, following that line, that the corrections neglected are numerically important, and even modify the parametric form of the final result, by removing any logarithms. Note that a somewhat related problem has been addressed in the context of the electroweak theory at finite temperatures [32], although for operators which are bilinear in the Higgs field rather than in fermionic fields.

The plan of this paper is the following. The basic notation used throughout is fixed in section 2. Some well-known properties of mesonic correlators at very high temperatures are reviewed in section 3. In section 4 we discuss how the mesonic correlators can be incorporated into the general setting of describing high-temperature QCD by a dimensionally reduced effective field theory [31]. The quark part of the lagrangian entering will be called three-dimensional (3d) "Non-Relativistic QCD", NRQCD$_3$. The basic matching computation between the original QCD and NRQCD$_3$ is carried out in section 5, and the first non-trivial correction to the correlation length of flavour non-singlet mesons is then determined from NRQCD$_3$ in section 6. In section 7 we compare with lattice results. Flavour singlet observables, for which the pattern is qualitatively different from the non-singlets, since they may mix with purely gluonic operators, are treated in section 8, and we conclude in section 9.

2. Notation

The theory we consider in this paper is QCD at a finite temperature T, with the euclidean quark lagrangian

$$\mathcal{L}_E^\psi = \bar\psi(\gamma_\mu D_\mu + M)\psi\,, \tag{2.1}$$

where $D_\mu = \partial_\mu - igA_\mu$, $A_\mu = A_\mu^c T^c$, T^c are hermitean generators of $\mathrm{SU}(N_c)$ normalised such that $\mathrm{Tr}\, T^c T^d = \frac{1}{2}\delta^{cd}$, and M is the mass matrix. For simplicity, we take M to be diagonal and degenerate, $M = \mathrm{diag}(m,\ldots,m)$, and in most of what follows, $m = 0$. The number of flavours appearing in ψ is denoted by N_f.

Given $\bar\psi,\psi$ we may define, as usual, various bilinears. The $N_f \times N_f$ flavour basis is generated by

$$F^a \equiv \{F^s, F^n\}\,, \qquad F^s \equiv 1_{N_f\times N_f}\,, \qquad n = 1,\ldots,N_f^2-1\,, \tag{2.2}$$

where $F^s = 1_{N_f\times N_f}$ is an $N_f \times N_f$ identity matrix ("singlet"), and the traceless F^n ("non-singlet") are assumed normalised such that

$$\mathrm{Tr}[F^m F^n] = \frac{1}{2}\delta^{mn}\,. \tag{2.3}$$

We may then consider scalar, pseudoscalar, vector, and axial vector objects,

$$S^a \equiv \bar\psi F^a \psi\,, \tag{2.4}$$
$$P^a \equiv \bar\psi \gamma_5 F^a \psi\,, \tag{2.5}$$
$$V^a_\mu \equiv \bar\psi \gamma_\mu F^a \psi\,, \tag{2.6}$$
$$A^a_\mu \equiv \bar\psi \gamma_\mu \gamma_5 F^a \psi\,. \tag{2.7}$$

Operator	Particle	Behaviour in free theory	Physical interpretation
S^s	f_0/σ	$q^2 B_{3\mathrm{d}}(2p_0)$	
S^n	a_0/δ	$q^2 B_{3\mathrm{d}}(2p_0)$	
P^s	η'	$q^2 B_{3\mathrm{d}}(2p_0)$	
P^n	π, η	$q^2 B_{3\mathrm{d}}(2p_0)$	
V_0^s	ω	$(q^2 + 4p_0^2) B_{3\mathrm{d}}(2p_0)$	baryon density
$V_\perp^s$		$(q^2 - 4p_0^2) B_{3\mathrm{d}}(2p_0)$	
V_0^n	ρ, ϕ	$(q^2 + 4p_0^2) B_{3\mathrm{d}}(2p_0)$	charge density
$V_\perp^n$		$(q^2 - 4p_0^2) B_{3\mathrm{d}}(2p_0)$	
A_0^s	f_1	$(q^2 + 4p_0^2) B_{3\mathrm{d}}(2p_0)$	
$A_\perp^s$		$(q^2 - 4p_0^2) B_{3\mathrm{d}}(2p_0)$	
A_0^n	a_1	$(q^2 + 4p_0^2) B_{3\mathrm{d}}(2p_0)$	
$A_\perp^n$		$(q^2 - 4p_0^2) B_{3\mathrm{d}}(2p_0)$	

Table 1: Different particle assignments and the nature of the singularity structure for the mesonic correlators considered (cf. section 3). The physical interpretation of V_0^n applies for $N_f = 3$. The notation $(\cdots)_\perp$ refers to polarizations transverse with respect to the spatial direction in which the correlation is measured; the longitudinal correlators all vanish in perturbation theory, due to current conservation in the chiral limit.

The operators just defined may be interpreted either directly as physical condensates or currents, or as interpolating operators for certain particle states. In three-flavour QCD ($N_f = 3$) with a degenerate mass matrix, for instance, some possible assignments are shown in table 1.

Given such operators, the correlators to be considered are of the form

$$C_{\boldsymbol{q}}[O^a, O^b] \equiv \int_0^{1/T} \mathrm{d}\tau \int \mathrm{d}^3 x \, e^{i\boldsymbol{q}\cdot\boldsymbol{x}} \, \langle O^a(\tau, \boldsymbol{x}) O^b(0, \mathbf{0}) \rangle \, . \tag{2.8}$$

The corresponding configuration space correlator is

$$C_{\boldsymbol{x}}[O^a, O^b] \equiv \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \, e^{-i\boldsymbol{q}\cdot\boldsymbol{x}} \, C_{\boldsymbol{q}}[O^a, O^b] = \int_0^{1/T} \mathrm{d}\tau \, \langle O^a(\tau, \boldsymbol{x}) O^b(0, \mathbf{0}) \rangle \, . \tag{2.9}$$

The expectation values are taken in a volume which is infinite in the spatial directions but finite in the temporal direction, with extent $1/T$. Gauge fields are assumed to obey periodic, fermions anti-periodic boundary conditions around the temporal direction.

The general structures of the correlators so defined are, because of rotational invariance,

$$C_{\boldsymbol{q}}[S^a, S^b] \, , C_{\boldsymbol{q}}[V_0^a, V_0^b] = \delta^{ab} f(q^2) \, , \tag{2.10}$$

$$C_{\boldsymbol{q}}[V_i^a, V_j^b] = \delta^{ab} \left[\left(\delta_{ij} - \frac{q_i q_j}{q^2} \right) t(q^2) + \frac{q_i q_j}{q^2} l(q^2) \right] , \tag{2.11}$$

and correspondingly with $S \to P$, $V \to A$. Here $q \equiv |\boldsymbol{q}|$. For conserved currents, $l(q^2) = 0$.

We expect, in general, that in the full theory the analytic structures of the functions $f(q^2), t(q^2), l(q^2)$ around the origin are characterised by simple poles. The reason is that, as we will recall presently, these functions can after analytic continuation be viewed as propagators of physical states in a (2+1)-dimensional confining theory. The spectrum of the confining theory consists of bound states, represented by poles. Consequently, the Fourier transforms should show exponential falloff at large distances. The purpose of this paper is to determine the pole locations or, equivalently, the long-distance exponential falloffs of the corresponding configuration space correlation functions, to order $\mathcal{O}(g^2)$. The coefficients of the exponential falloffs are often referred to as screening masses.

3. Correlators at very high temperatures

To start with, it may be useful to recall the behaviour of the correlation functions we are interested in at very high temperatures. In this limit, asymptotic freedom should guarantee that we can use perturbation theory, and the correlators can be computed with free fermions (figure 1a). One finds, in dimensional regularisation,

$$C_{\boldsymbol{q}}[O^a, O^b] = \mathrm{Tr}[F^a F^b]\, N_c T \sum_{n=-\infty}^{\infty} \int \frac{\mathrm{d}^{3-2\epsilon} p}{(2\pi)^{3-2\epsilon}} \frac{1}{[p_n^2 + \boldsymbol{p}^2][p_n^2 + (\boldsymbol{p}+\boldsymbol{q})^2]} \mathrm{Tr}\Big[(\not{p} + \not{q})\, \Gamma^a \not{p}\, \Gamma^b\Big]\,, \tag{3.1}$$

where Γ^a is the Dirac matrix appearing in O^a, $\Gamma^a = \{1, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5\}$, $\not{p} \equiv \gamma_\mu p_\mu$, and p_n denotes the fermionic Matsubara frequencies, $p_n \equiv 2\pi T(n+1/2)$. The Dirac algebra and the integration are trivially carried out: the result contains either constants, which correspond to contact terms $\sim \delta(\boldsymbol{x})$ after going into configuration space according to eq. (2.9), or the function

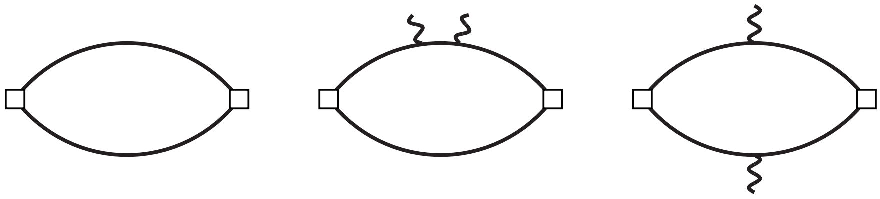
$$B_{3\mathrm{d}}(2p_n) \equiv \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{[p_n^2 + \boldsymbol{p}^2][p_n^2 + (\boldsymbol{p}+\boldsymbol{q})^2]} = \frac{i}{8\pi q} \ln \frac{2p_n - iq}{2p_n + iq}\,. \tag{3.2}$$

The qualitative functional forms of the terms including this function for the different O^a are shown in table 1, for $p_n = p_0$.

The expression in eq. (3.1) contains a sum over all $p_n = 2\pi T(n + 1/2)$. Since the non-trivial structure appears at the point $2p_n$ (cf. eq. (3.2)), the correlator is dominated at large distances (small q) by the smallest Matsubara frequencies, $\pm p_0 = \pm\pi T$. These frequency modes live in three spatial dimensions. If we now imagine viewing the three-dimensional theory rather as a (2+1)-dimensional one, and call the direction in which the correlation is measured, the time t, then the screening mass corresponds to the energy of a state consisting of two on-shell massive quarks, each of "mass" p_0. Put in this language, our problem will be to compute the first correction to the energy of the two-particle state.

We may also note that the nature of the singularity at $q = \pm 2ip_0$ in eq. (3.2) is a branch cut. In a (2+1)-dimensional world, this just corresponds to an energy threshold for producing two free on-shell particles of "mass" p_0. We may expect this structure to get qualitatively modified by the interactions, which place the quarks off-shell and bind them together, as just described: the threshold should then convert to a pole. The computation of the functional form of the singularity is beyond the scope of the present paper, however: we simply study how the location of the threshold at $2p_0$ gets modified.

Note from table 1 that for the charge-charge correlators (V_0^a, A_0^a), there is a prefactor $(q^2 + 4p_0^2)$ which vanishes at the point of the branch singularity in $B_{3\mathrm{d}}(2p_0)$. This means



(a) (b) (c)

Figure 1: (a): The leading order correlator. (b), (c): Graphs representing non-trivial gluon condensates in the operator product expansion for the mesonic correlation function.

that a contribution of the singularity at $q = \pm 2ip_0$ is suppressed, and the correlator tends to decay even faster: it is easy to see that in configuration space, it is power-suppressed in $1/p_0 t$. The issue is even clearer if, for a moment, one imagines going to a world with one space dimension only: then

$$B_{\rm 1d}(2p_0) = \int \frac{{\rm d}p}{2\pi} \frac{1}{[p_0^2 + p^2][p_0^2 + (p+q)^2]} = \frac{1}{p_0} \frac{1}{q^2 + 4p_0^2} \,, \tag{3.3}$$

and the singularity (a real pole in this case) would be completely cancelled by the prefactor, leaving only (short-distance) contact terms.[1] We will return to this issue below.

To conclude this section, we briefly inspect the correlators from yet another angle. Let us, for a moment, consider the correlator not around the pole at $q = \pm 2ip_0$, but for a very large (real) q. Then the correlator can be computed in the operator product expansion: taking a background of constant (zero Matsubara mode) gauge fields (figures 1b, 1c) one obtains, for instance,

$$C_{\boldsymbol{q}}[S^a, S^b] \approx {\rm Tr}[F^a F^b]\, T \left[-\frac{N_c}{4\pi} \left(4p_0 + iq \ln \frac{2p_0 - iq}{2p_0 + iq} \right) + \frac{1}{\pi p_0} \frac{(q^2)^2}{(q^2 + 4p_0^2)^2} g^2 \,{\rm Tr}\, A_0^2 + \cdots \right] . \tag{3.4}$$

Extrapolating then back towards the threshold region, say $q \sim 2ip_0 + \mathcal{O}(g^2 T)$, one sees how the corrections related to the gauge fields are becoming increasingly important. In order to systematically evaluate them we have to turn, however, to other methods.

4. Basic structure of NRQCD$_3$

To go beyond the accuracy of figure 1a, we need to compute graphs of the qualitative types in figure 2. Just the graphs shown are not enough, however: one also needs to account for various kinds of iterations of these basic topologies. The reason is that near the threshold, quarks can be almost on-shell, $|1/\not{p}| \sim \mathcal{O}(1/g^2 T)$, and thus compensate for the appearance of $\mathcal{O}(g^2 T)$ in the vertices. Therefore, there is a large set of graphs which has to be identified and resummed, in order to obtain the first correction to the threshold location.

As in many other contexts, a convenient organising principle for carrying out such resummations is offered by effective field theory methods. As already explained, in the present case we can view the correlation lengths as (2+1)-dimensional bound states of

[1]This remains true even after inclusion of the other p_n's.

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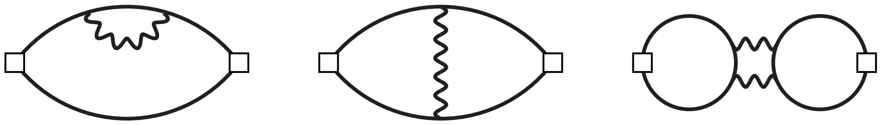

Figure 2: Schematic representations of various classes of beyond-the-leading order corrections: (a) quark self-energy correction; (b) quark-antiquark interaction via gluon exchange; (c) for flavour singlets, the correlation can also be mediated by purely gluonic states.

heavy particles of "mass" p_0, much larger than the infrared scales $\sim gT, g^2T$ of gauge field dynamics at high temperatures [3]. The masses of such heavy bound states can be determined, like those of quarkonia at $T = 0$, through an effective theory called the non-relativistic QCD (NRQCD) [33]. Our case does lead to some significant differences with respect to this framework; for instance, that the "masses", p_0, are not really masses in the conventional sense, but conserve chiral invariance and are directly scale invariant physical quantities. Despite the differences, we will refer to the general framework to be used as NRQCD$_3$. The philosophy of applying NRQCD techniques to the present problem was introduced by Huang and Lissia [31]. We start by showing how the non-relativistic structure emerges on the tree-level.

4.1 Tree-level

As seen in eq. (3.1), at leading order the correlators considered get independent contributions from the various quark Matsubara modes, with the lowest ones being dominant at large distances. Let us therefore again concentrate on $p_0 \equiv \pi T$. The quark lagrangian for this Matsubara mode is

$$\mathcal{L}_E^\psi = \bar{\psi}[i\gamma_0 p_0 - ig\gamma_0 A_0 + \gamma_k D_k + \gamma_3 D_3]\psi \,, \tag{4.1}$$

where $k \equiv 1, 2$.

In eq. (4.1), ψ interacts with gluonic zero Matsubara modes only. The lagrangian describing their dynamics, obtained after integrating out all quarks and the non-zero Matsubara modes of gluons, is the dimensionally reduced lagrangian, of the form [3]

$$\mathcal{L}_E^A = \frac{1}{2}\,\mathrm{Tr}\, F_{ij}^2 + \mathrm{Tr}[D_i, A_0]^2 + m_\mathrm{E}^2\,\mathrm{Tr}\, A_0^2 + \lambda_\mathrm{E}^{(1)}(\mathrm{Tr}\, A_0^2)^2 + \lambda_\mathrm{E}^{(2)}\,\mathrm{Tr}\, A_0^4 + \cdots \,. \tag{4.2}$$

Here $i = 1, \ldots, 3$, $F_{ij} = (i/g_\mathrm{E})[D_i, D_j]$, and $D_i = \partial_i - ig_\mathrm{E} A_i$, where $A_\mu = A_\mu^c T^c$ and the fields have now the dimension GeV$^{1/2}$, after a trivial rescaling with $T^{1/2}$. The parameters, to the orders that we need them, are

$$m_\mathrm{E}^2 = \left(\frac{N_c}{3} + \frac{N_f}{6}\right) g^2 T^2 \,, \qquad g_\mathrm{E}^2 = g^2 T \,, \tag{4.3}$$

while $\lambda_\mathrm{E}^{(i)} = \mathcal{O}(g^4 T)$ can be ignored. The parameters have been determined up to next-to-leading order [34, 11, 35]. According to eqs. (4.2), (4.3), infrared gauge field dynamics is sensitive to two scales: the perturbative scale $m_\mathrm{E} \sim gT$ associated with the colour-electric component A_0, and the (2+1)-dimensional confinement scale $g_\mathrm{E}^2 \sim g^2 T$ associated with the colour-magnetic components A_i.

Let us next set up some notation. In the following, we use rotational invariance to choose the vector $\boldsymbol{q}$ in eq. (2.8) to point in the x_3-direction. According to eq. (2.9) we are thus effectively considering correlators averaged over the (x_1, x_2)-plane, and depending on x_3. We will often denote the euclidean x_3-coordinate by t, and the vector (x_1, x_2) by $\boldsymbol{x}_\perp$. The corresponding Fourier modes are $p_3, \boldsymbol{p}_\perp$. The indices k, l are assumed to have the values $1, 2$, and label the transverse directions. Given this choice of $\boldsymbol{q}$ we can consider, without loss of generality, the correlator

$$C_t[O^a, O^b] \equiv \int \frac{\mathrm{d}q_3}{2\pi} e^{-iq_3 t}\, C_{(0,0,q_3)}[O^a, O^b] = \int_0^{1/T} \mathrm{d}\tau \int \mathrm{d}^2\boldsymbol{x}_\perp \, \langle O^a(\tau, \boldsymbol{x}_\perp, t) O^b(0, \boldsymbol{0}_\perp, 0) \rangle \,, \tag{4.4}$$

rather than $C_{\boldsymbol{x}}[O^a, O^b]$ in eq. (2.9).

Once this choice has been made, the quark lagrangian in eq. (4.1) can be written in a suggestive form. Let us make a basis transformation from the standard representation

$$\gamma_0 = \begin{pmatrix} 1_{2\times 2} & 0 \\ 0 & -1_{2\times 2} \end{pmatrix}, \qquad \gamma_i = \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} 0 & 1_{2\times 2} \\ 1_{2\times 2} & 0 \end{pmatrix}, \tag{4.5}$$

where σ_i are the Pauli matrices, to a new basis, $\gamma_\mu^{\rm new} = U \gamma_\mu^{\rm standard} U^{-1}$, with

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}. \tag{4.6}$$

In this basis,

$$\gamma_0 \not{p} = \begin{pmatrix} (p_0 + ip_3) 1_{2\times 2} & -\epsilon_{kl} p_k \sigma_l \\ \epsilon_{kl} p_k \sigma_l & (p_0 - ip_3) 1_{2\times 2} \end{pmatrix}, \tag{4.7}$$

where ϵ_{kl} is antisymmetric and $\epsilon_{12} = 1$. Thus, denoting

$$\psi \equiv \begin{pmatrix} \chi \\ \phi \end{pmatrix}, \tag{4.8}$$

where χ, ϕ are two-component spinors, the lagrangian in eq. (4.1) becomes

$$\mathcal{L}_E^\psi = i\chi^\dagger \Big[p_0 - gA_0 + D_3 \Big] \chi + i\phi^\dagger \Big[p_0 - gA_0 - D_3 \Big] \phi + \phi^\dagger \epsilon_{kl} D_k \sigma_l \chi - \chi^\dagger \epsilon_{kl} D_k \sigma_l \phi \,. \tag{4.9}$$

Leaving out, for brevity, the flavour structures, the bilinear operators of eqs. (2.4)–(2.7) become

$$S = \chi^\dagger\phi + \phi^\dagger\chi \,, \tag{4.10}$$
$$P = \chi^\dagger\sigma_3\phi - \phi^\dagger\sigma_3\chi \,, \tag{4.11}$$
$$V_0 = \chi^\dagger\chi + \phi^\dagger\phi \,, \tag{4.12}$$
$$V_k = -\epsilon_{kl}(\chi^\dagger\sigma_l\phi - \phi^\dagger\sigma_l\chi) \,, \tag{4.13}$$
$$V_3 = \chi^\dagger\chi - \phi^\dagger\phi \,, \tag{4.14}$$
$$A_0 = \phi^\dagger\sigma_3\phi - \chi^\dagger\sigma_3\chi \,, \tag{4.15}$$
$$A_k = -i(\chi^\dagger\sigma_k\phi + \phi^\dagger\sigma_k\chi) \,, \tag{4.16}$$
$$A_3 = -i(\chi^\dagger\sigma_3\chi + \phi^\dagger\sigma_3\phi) \,. \tag{4.17}$$

Note that the parity transformation, $\psi(\tau, \boldsymbol{x}) \to \gamma_0 \psi(\tau, -\boldsymbol{x})$, reads $\chi(\tau, \boldsymbol{x}) \to \phi(\tau, -\boldsymbol{x})$, $\phi(\tau, \boldsymbol{x}) \to \chi(\tau, -\boldsymbol{x})$ in this basis.

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In section 3, we saw that in the correlators we are interested in, the quarks are almost "on-shell". Let us therefore consider eq. (4.9) for on-shell configurations, in momentum space. We fix $\boldsymbol{p}_\perp$ and consider the functional dependence on p_3. In the free case, the on-shell point is at $p_0^2+p^2=0$, i.e., $p_3 = \pm i[p_0 + \boldsymbol{p}_\perp^2/(2p_0)+\cdots]$. The positive and negative "energy" solutions could be called the particles and the anti-particles.

Because the quarks interact (at the tree-level) with bosonic Matsubara zero modes only, we expect that off-shellness is related to the momentum scales of the latter: $|p_3 \pm ip_0| \lesssim gT$. Assuming this to be the case, the idea is to construct an effective theory describing the soft scale dynamics. For instance, consider the pole around $p_3 \approx ip_0$. Then χ represents the light mode, while ϕ is heavy. Being so far at the classical level, we can solve for ϕ from the equations of motion, and substitute the solution back to the lagrangian, resulting in an action for χ alone (cf., e.g., the pedagogic presentation in [36] for the 4d zero temperature case). The same treatment naturally applies to ϕ around its pole. Expanding finally in powers of $1/p_0$, we obtain a "diagonalised" on-shell effective lagrangian for two independent light modes, with a non-relativistic structure:

$$\begin{aligned}\mathcal{L}_E^\psi \approx\; & i\chi^\dagger\left[p_0 - gA_0 + D_3 - \frac{1}{2p_0}\left(D_k^2 + \frac{g}{4i}[\sigma_k,\sigma_l]F_{kl}\right)\right]\chi + \\ & + i\phi^\dagger\left[p_0 - gA_0 - D_3 - \frac{1}{2p_0}\left(D_k^2 + \frac{g}{4i}[\sigma_k,\sigma_l]F_{kl}\right)\right]\phi + \mathcal{O}\left(\frac{1}{p_0^2}\right). \end{aligned} \tag{4.18}$$

The mesonic states are most usefully still represented by the operators in eqs. (4.10)–(4.17).

For future reference, let us note that the free propagators following from eq. (4.18) are

$$\langle \chi_u(p)\chi_v^*(q)\rangle = \delta_{uv}(2\pi)^3\delta^{(3)}(p-q)\frac{-i}{M + ip_3 + \boldsymbol{p}_\perp^2/(2p_0)}, \tag{4.19}$$

$$\langle \phi_u(p)\phi_v^*(q)\rangle = \delta_{uv}(2\pi)^3\delta^{(3)}(p-q)\frac{-i}{M - ip_3 + \boldsymbol{p}_\perp^2/(2p_0)}, \tag{4.20}$$

where $M \equiv p_0$, and u, v contain the spinor, flavour and colour indices. We may also remark that going into configuration space, the propagators become

$$\langle \chi_u(x)\chi_v^*(y)\rangle = -i\delta_{uv}\theta(x_3-y_3)\frac{p_0}{2\pi|x_3-y_3|}\exp\left(-M|x_3-y_3| - \frac{p_0(\boldsymbol{x}_\perp - \boldsymbol{y}_\perp)^2}{2|x_3-y_3|}\right), \tag{4.21}$$

$$\langle \phi_u(x)\phi_v^*(y)\rangle = -i\delta_{uv}\theta(y_3-x_3)\frac{p_0}{2\pi|x_3-y_3|}\exp\left(-M|x_3-y_3| - \frac{p_0(\boldsymbol{x}_\perp - \boldsymbol{y}_\perp)^2}{2|x_3-y_3|}\right). \tag{4.22}$$

This implies forward time propagation for χ and backward for ϕ. For $p_0 \to \infty$,

$$\frac{p_0}{2\pi|x_3-y_3|}\exp\left(-\frac{p_0(\boldsymbol{x}_\perp - \boldsymbol{y}_\perp)^2}{2|x_3-y_3|}\right) \to \delta^{(2)}(\boldsymbol{x}_\perp - \boldsymbol{y}_\perp), \tag{4.23}$$

indicating that the quarks become static.

To conclude this section, it may be noted that in eqs. (4.10)–(4.17) the charges (V_0, A_0) and the third components of the currents (V_3, A_3) are of the type $\sim \chi^\dagger\chi + \phi^\dagger\phi$. This implies that the corresponding correlators decay very rapidly: according to eqs. (4.21), (4.22), they in fact vanish identically at non-zero distances, being of the form $\sim \theta(t)\theta(-t)$. For the third components of the currents this is due to current conservation (the correlator is a constant, and vanishes in infinite volume, up to possible contact terms), while for the charges, this is related to the discussion in section 3, where we observed the prefactor $(q^2 + 4p_0^2)$ in the free correlators. In the on-shell limit, this causes the correlators to vanish; in reality, they are power-suppressed in $1/p_0 t$. In the following, we will concentrate on the correlators which decay more slowly, that is, which are represented by operators of the type $\sim \chi^\dagger\phi + \phi^\dagger\chi$.

4.2 Power-counting and loop corrections

The construction of the effective theory in eq. (4.18) was so far on the tree-level. In order to be sure that the IR structure of the effective theory is really that of finite temperature QCD, and in order to determine the radiative corrections we are ultimately interested in, we need to consider loop corrections. In particular, we should find out the order to which one needs to go in the matching computation between the original finite temperature QCD and NRQCD$_3$, as well as the resolution with which the dynamics inside NRQCD$_3$ needs to be treated, given that we are interested in the $\mathcal{O}(g^2)$ correction to the spatial mesonic correlation lengths. For these purposes, it is useful to set up some parametric power-counting rules.

Since the general upshot will be rather obvious, let us state it immediately: in order to determine the $\mathcal{O}(g^2)$ correction to the mesonic correlation lengths, the additive parameter p_0 in eq. (4.18) becomes a matching coefficient (to be denoted by M, as in eqs. (4.19)–(4.22)) which has to be determined to 1-loop order. On the other hand, no other parameters need to be matched at this level, and some interactions can even be dropped out: the final form of the theory is displayed in eq. (4.25) below. This matching takes care of the "hard part" in loop integrals of the type in figure 2a, while the dynamics within the effective theory accounts for diagrams of the type in figure 2b, where the gauge fields are always soft.

To account for the full correlator defined in eq. (2.8), one should also carry out operator matching to order $\mathcal{O}(g^2)$. We will not need to consider it here, however, since we will not attempt to determine the overall magnitude of the correlator, only the coefficient of its exponential falloff.

Let us now turn to the power-counting rules. As mentioned, the parametric scales in the problem are $\sim \pi T, gT, g^2T$. Let us write $m \equiv \pi T, v \equiv g$, and denote by Δp_3 the off-shellness of the "energy", $\Delta p_3 = p_3 \pm ip_0$. Following the terminology of NRQCD, one can define four different regions of phase space:

$$\begin{aligned} \text{hard (h)}: &\quad \Delta p_3 \sim |\boldsymbol{p}_\perp| \sim m\,, \\ \text{soft (s)}: &\quad \Delta p_3 \sim |\boldsymbol{p}_\perp| \sim mv\,, \\ \text{potential (p)}: &\quad \Delta p_3 \sim mv^2\,, |\boldsymbol{p}_\perp| \sim mv\,, \\ \text{ultrasoft (us)}: &\quad \Delta p_3 \sim |\boldsymbol{p}_\perp| \sim mv^2\,. \end{aligned} \tag{4.24}$$

By definition, the NRQCD$_3$ theory in eqs. (4.18), (4.2) is chosen to contain at most soft quarks and gluons as dynamical degrees of freedom. In fact, close to on-shell, the quarks of NRQCD$_3$ are potential, since $\Delta p_3 \sim |\boldsymbol{p}_\perp^2|/m \sim mv^2$.

In order to estimate the importance of the various momentum regions in eq. (4.24), it is helpful to assign power-counting rules also to the corresponding field components. This can be done by going to configuration space and requiring that the action, a pure number, is parametrically of order unity. For soft and potential quarks, $\chi_\mathrm{s}, \chi_\mathrm{p}$, it follows from $S \sim \int \mathrm{d}t\, \mathrm{d}^2\boldsymbol{x}_\perp\, \chi^\dagger i\partial_t \chi \sim 1$ that $\chi \sim 1/|\boldsymbol{x}_\perp| \sim mv$. For soft gluons A_s, $S \sim \int \mathrm{d}t\, \mathrm{d}^2\boldsymbol{x}_\perp\, A_\mathrm{s}\partial_t^2 A_\mathrm{s} \sim 1$ implies that $A_\mathrm{s} \sim (t/\boldsymbol{x}_\perp^2)^{1/2} \sim m^{1/2}v^{1/2}$. For ultrasoft gluons A_us, correspondingly, $A_\mathrm{us} \sim (t/\boldsymbol{x}_\perp^2)^{1/2} \sim m^{1/2}v$. In addition, when operating on on-shell quarks, the time derivative can be estimated as $\partial_t \sim 1/t \sim mv^2$.

Given these rules and that $g_\mathrm{E}^2 \sim mv^2$, we note that $g_\mathrm{E}A_\mathrm{s} \sim mv^{3/2}$, $g_\mathrm{E}A_\mathrm{us} \sim mv^2$. The important conclusion follows that soft and ultrasoft gauge fields are parametrically of *higher order* than the transverse momentum $\nabla_\perp \sim mv$. In particular, given that we are interested in quarks which are off-shell by at most $\mathcal{O}(mv^2) \sim \mathcal{O}(g^2T)$, we should note that this magnitude is already saturated by $\nabla_\perp^2/m$: no transverse covariant derivatives are needed in the quark action! On the contrary, the gauge fields *are* significant in relation to ∂_t.

To summarise, if we want to keep the lagrangian up to and including the order $\mathcal{O}(m^3v^4)$, such that we know the on-shell quark self-energy up to order $\mathcal{O}(mv^2)$, it is enough to replace eq. (4.18) with the action

$$\mathcal{L}_E^\psi = i\chi^\dagger\left(M - g_\mathrm{E}A_0 + D_t - \frac{\nabla_\perp^2}{2p_0}\right)\chi + i\phi^\dagger\left(M - g_\mathrm{E}A_0 - D_t - \frac{\nabla_\perp^2}{2p_0}\right)\phi\,, \tag{4.25}$$

where $D_t \equiv D_3 = \partial_3 - ig_\mathrm{E}A_3$, and the gauge fields have now the same 3d normalisation as in eq. (4.2). Furthermore, the only parameter requiring a matching computation is M.

To conclude this section, let us briefly view the dynamics of the remaining degrees of freedom. Since the quarks are very heavy, of mass p_0, we may compute the static potential in (2+1) dimensions. Its structure is,

$$V(r) \sim g_\mathrm{E}^2 \ln r + g_\mathrm{E}^4 r + \mathcal{O}(g_\mathrm{E}^6 r^2)\,. \tag{4.26}$$

If inserted into a Schrödinger equation, we get

$$\frac{1}{p_0}\frac{\partial^2}{\partial r^2} \sim V(r) \sim g_\mathrm{E}^2 \ln r\,, \tag{4.27}$$

and therefore $|\boldsymbol{p}_\perp| \sim \partial/\partial r \sim 1/r \sim \sqrt{g_\mathrm{E}^2 p_0} \sim gT$. We see that the expansion parameter related to the static potential is $\mathcal{O}(g_\mathrm{E}^2 r) \sim \mathcal{O}(g)$. Consequently, for the correction of order $E \sim |\boldsymbol{p}_\perp|^2/p_0 \sim g^2T$, it is sufficient to compute the static potential to leading order. The leading order contribution itself may, however, already be infrared divergent, in the sense that the scale appearing inside the logarithm in eq. (4.26) remains to be determined.

5. Matching from QCD to NRQCD$_3$

As explained above, the parameter we need to determine through a matching computation is M, as defined by eq. (4.25). This computation was not carried out in [31], although many other matchings related to NRQCD$_3$ were.

In order to avoid having to deal with any wave function normalisation factors, we will carry out the matching by computing the quark "pole mass", or finite temperature euclidean dispersion relation, both in the original finite temperature QCD, as well as with the theory of eq. (4.25). Both computations produce a gauge-invariant result, and requiring the outcomes to be equivalent, leads to an expression for M in terms of the parameters of finite temperature QCD.

In order to monitor explicitly the gauge parameter dependence and the IR sensitivity of the computations carried out, we take the gluon propagator to be of the form

$$\langle A^a_\mu(p) A^b_\nu(q) \rangle = \delta^{ab} (2\pi)^{4-2\epsilon} \delta^{(4-2\epsilon)}(p+q) \left[\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \frac{1}{p^2+\lambda^2} + \frac{p_\mu p_\nu}{p^2} \frac{\xi}{p^2+\xi\lambda^2} \right] . \quad (5.1)$$

That is, we introduce a gauge parameter ξ as well as an IR-cutoff λ. As we will see, the pole masses, computed in both ways, are independent of ξ, and allow to take the limit $\lambda \to 0$. The same is then also true for their difference, determining the matching coefficient we are interested in.

An important and somewhat non-trivial point is the way the computation is to be carried out on the side of NRQCD_3. The "problem" is that we want to use dimensional regularisation, yet obey the power-counting conventions as discussed above. If one, however, takes the free propagators as determined by eq. (4.25), shown in eqs. (4.19), (4.20), then (irrespective of whether or not one keeps M there explicitly or shifts it away by modifying the integration contour for p_3) momenta of order $|\boldsymbol{p}_\perp| \sim p_0$ do contribute. This, however, we do not want, because our power-counting rules assumed the dynamical scales inside the effective theory to be at most soft, $|\boldsymbol{p}_\perp| \lesssim mv$, while $p_0 \sim m$. The way to avoid the problem with dimensional regularisation has been clarified in [37]: one should carry out the matching in the soft region of the phase space, $p_3 \sim |\boldsymbol{p}_\perp| \sim mv$, where the transverse part $\boldsymbol{p}_\perp^2/p_0$ can be treated as a perturbation. That is, we match the operators of NRQCD_3 order by order in $1/p_0$.

The situation would be different if we were to treat NRQCD_3 also on the lattice. In order to match from dimensional regularisation to lattice regularisation, say, one would then really have to compute using the same forms of propagators in both schemes. This leads to additional logarithmic ultraviolet divergences in the parameter M, needed in order to cancel the corresponding divergences appearing in the dynamics within NRQCD_3. Carrying out the computation this way is not much more complicated than what we do here, but since the results are not needed for our present purposes, we do not display them.

On the side of QCD, the inverse propagator evaluated at the position of the tree-level pole, $p^2 = 0$, reads

$$\Sigma(p) = i\not{p} - ig^2 C_F \sum\!\!\!\!\!\!\!\int_q \frac{\gamma_\mu (\not{p} - \not{q}) \gamma_\mu}{(p-q)^2_f (q^2+\lambda^2)_b} \bigg|_{p^2=0} , \quad (5.2)$$

where $(\cdots)_f$ denotes fermionic Matsubara modes, $(\cdots)_b$ bosonic, and $C_F = (N_c^2-1)/2N_c$. The result comes from the transverse part of the gluon propagator; the longitudinal gauge parameter dependent part does not contribute at the pole. The task is now to find the zeros of $\Sigma(p)$, to relative order $\mathcal{O}(g^2)$.

In order to streamline the task, we may set $\boldsymbol{p}_\perp = 0$, and multiply $\Sigma(p)$ from the left with γ_0. The result obtains a simple form if we choose the basis of γ-matrices introduced in section 4.1. Indeed, off-diagonal terms vanish after integration over q, and the result becomes block-diagonal. Moreover, the upper two components are degenerate, as well as the lower two components, and the latter follow from the former simply by $(p_3 \to -p_3, q_3 \to -q_3)$. Therefore it is enough to consider

$$\begin{aligned}\Big[\gamma_0\Sigma(p)\Big]_{11} = i\Big\{p_0 + ip_3 - g^2 C_F \sum\!\!\!\!\!\!\!\int_q \frac{1}{(p-q)_f^2(q^2+\lambda^2)_b}\Big[&+p_0 - ip_3 - q_0 + iq_3 - \quad (A_0) \\ &- p_0 + ip_3 + q_0 - iq_3 - \quad (A_3) \\ &- 2p_0 - 2ip_3 + 2q_0 + 2iq_3 \quad (A_k)\Big]\Big\}\,,\end{aligned} \tag{5.3}$$

where the gauge field components leading to the various contributions have also been indicated, and we have set the spatial dimensionality to $d = 3$, anticipating the fact that the result is UV-finite. The $\mathcal{O}(g^2)$ term is to be evaluated at the tree-level pole position $p_0 = -ip_3$.

It can be observed from eq. (5.3) that both A_0 and A_3 give possibly logarithmically IR-divergent integrals (if $\lambda \to 0$), from the terms with $p_0 - ip_3$ in the numerator, but the sum is finite since they come with opposite signs. For A_k, on the other hand, the contribution involving p_0, ip_3 comes in the combination $p_0 + ip_3$, and vanishes altogether at the pole. The correction as a whole is therefore IR finite, and can be evaluated with $\lambda \to 0$, whereby the integral becomes

$$\sum\!\!\!\!\!\!\!\int_q \frac{2(q_0 + iq_3)}{(p-q)_f^2(q^2)_b}\bigg|_{p_0=-ip_3} = \frac{1}{p_0}\sum\!\!\!\!\!\!\!\int_q \frac{2p\cdot q}{(p-q)_f^2(q^2)_b}\bigg|_{p^2=0} = \frac{1}{p_0}\sum\!\!\!\!\!\!\!\int_q\Big[-\frac{1}{q_b^2} + \frac{1}{q_f^2}\Big] = -\frac{T^2}{8p_0}\,. \tag{5.4}$$

Consequently,

$$[\gamma_0\Sigma(p)]_{11} \approx i\left\{p_0 + ip_3 + g^2 C_F \frac{T^2}{8p_0}\right\}, \tag{5.5}$$

and the dispersion relation, or pole position, becomes

$$p_3 \approx i\left[p_0 + g^2 C_F \frac{T^2}{8p_0}\right]. \tag{5.6}$$

On the side of NRQCD$_3$ we proceed as explained above, following [37]. The gauge field propagator is treated as in eq. (5.1) at the current stage (during matching, any possible IR cutoffs should be the same in the two theories). The inverse propagator now becomes

$$\begin{aligned}\hat\Sigma(p) = i\Big\{M + ip_3 - g_{\rm E}^2 C_F \int \frac{{\rm d}^{3-2\epsilon}q}{(2\pi)^{3-2\epsilon}} \frac{1}{M + ip_3 - iq_3}\Big[&+\frac{1}{q^2+\lambda^2} - \quad (A_0) \\ &- \frac{1}{q^2+\lambda^2} \quad (A_3)\Big]\Big\}\,.\end{aligned} \tag{5.7}$$

The potential IR divergences from the two gauge field components are the same as in eq. (5.3), but they cancel, as they did there, such that the 1-loop correction is finite, and in fact vanishes. Thus, the dispersion relation is simply

$$p_3 \approx iM\,. \tag{5.8}$$

Matching now eqs. (5.6) and (5.8), we obtain

$$M = p_0 + g^2 T \frac{C_F}{8\pi} + \mathcal{O}(g^4 T)\,. \tag{5.9}$$

6. Perturbative solution within NRQCD$_3$

After the construction of the effective theory, which accounts for hard gluons in figure 2(a), we still have to account for the soft part of figure 2(a) as well as for all of figure 2(b), where the gluons can only be soft (i.e., Matsubara zero modes). This is to be done by computing the analogues of the correlators in eq. (4.4), using the effective theory in eqs. (4.2), (4.25), and the operators in eqs. (4.10)–(4.17). Note that because of the appearance of Debye screening (represented by the parameter m_E^2 in eq. (4.2)), the gauge field components A_0, A_3 behave now differently: the propagators are

$$\langle A_0^a(p) A_0^b(q)\rangle \;=\; \delta^{ab}(2\pi)^{3-2\epsilon}\delta^{(3-2\epsilon)}(p+q)\frac{1}{p^2+m_\mathrm{E}^2}\,, \tag{6.1}$$

$$\langle A_i^a(p) A_j^b(q)\rangle \;=\; \delta^{ab}(2\pi)^{3-2\epsilon}\delta^{(3-2\epsilon)}(p+q)\left[\left(\delta_{ij}-\frac{p_i p_j}{p^2}\right)\frac{1}{p^2+\lambda^2}+\frac{p_i p_j}{p^2}\frac{\xi}{p^2+\xi\lambda^2}\right]. \tag{6.2}$$

We have again introduced a fictitious infrared regulator λ, as well as a general gauge parameter ξ, just in order to check that the results are independent of them.

The correlators of eq. (4.4) take the form

$$C_t[O^a, O^b] \sim \int \mathrm{d}^2\boldsymbol{x}_\perp \left\langle O^a(\boldsymbol{x}_\perp, t) O^b(\boldsymbol{0}_\perp, 0)\right\rangle, \tag{6.3}$$

where we have suppressed the dependence on the τ-coordinate which disappeared as we moved to Matsubara frequency space. As already mentioned, we are ignoring corrections of relative order $\mathcal{O}(g^2)$ in the overall normalisation of eq. (6.3), but not in the coefficient of the exponential falloff.

Even though the operators in eqs. (4.10)–(4.17) contain two terms, $\sim \chi^\dagger\phi + \phi^\dagger\chi$, only one product contributes in eq. (6.3): this is the one where both quarks are forward propagating (cf. eqs. (4.21), (4.22)), $\sim \langle \phi^\dagger(t)\chi(t)\,\chi^\dagger(0)\phi(0)\rangle$. In other words, we need to determine the Green's function for the composite object $\int \mathrm{d}^2\boldsymbol{x}_\perp\, \phi_u^*(\boldsymbol{x}_\perp, t)\Gamma_{uv}\chi_v(\boldsymbol{x}_\perp, t)$, where Γ_{uv} is a 2×2 matrix in spinor space, an $N_f \times N_f$ matrix in flavour space, and an $N_c \times N_c$ matrix in colour space. Since the spinor and flavour indices play no role in the following, they will be suppressed.

As we have argued in section 4.2, it should be rather obvious how to proceed now: simply integrate out the gauge fields, to produce a static potential for the $\phi^*\chi$ pair, and solve the corresponding Schrödinger equation. The reason this works is that, as we will show *a posteriori*, the non-perturbative ultrasoft gauge fields do not contribute at the present order. The problem could also be formalised for instance by going through the PNRQCD effective theory [38, 39], from which the soft gluonic scales have been systematically integrated out. For our modest purposes here, involving only the Coulomb-type potential, going through the full formalism of PNRQCD seems somewhat too elaborate, however.

What we rather wish to do is to recall on a concrete but somewhat heuristic level how the Schrödinger equation can be seen to emerge, just in order to get all the signs right in our final expression.

The Green's function we need to consider is eq. (6.3). We would like to find the equation of motion obeyed by this Green's function, which we expect, at least at large t, to be of the form $(\partial_t - H)G(t) = C\delta(t)$, where H is a differential operator and C is a constant. Then the exponential decay is determined by the lowest eigenvalue of H.

In order to find H, we now define a point-splitting (which is essentially gauge-invariant because no spatial components of gauge fields appear in eq. (4.25)), by introducing a vector $\boldsymbol{r}$ and, with the intention of taking $\boldsymbol{r} \to 0$ afterwards, consider the modified correlator

$$C(\boldsymbol{r},t) \equiv \int \mathrm{d}^{2-2\epsilon}\boldsymbol{R} \left\langle \phi^* \left(\boldsymbol{R} + \frac{\boldsymbol{r}}{2}, t\right) \chi \left(\boldsymbol{R} - \frac{\boldsymbol{r}}{2}, t\right) \; \chi^*(\boldsymbol{0},0)\phi(\boldsymbol{0},0) \right\rangle . \tag{6.4}$$

We denote the tree and 1-loop contributions by $C^{(0)}(\boldsymbol{r},t), C^{(1)}(\boldsymbol{r},t)$. It is then easy to see that

$$\left[\partial_t + 2M - \frac{1}{p_0}\nabla^2_{\boldsymbol{r}}\right] C^{(0)}(\boldsymbol{r},t) \propto \delta(t)\delta^{(2-2\epsilon)}(\boldsymbol{r}) . \tag{6.5}$$

The result for the 1-loop contribution (a sum of the first two graphs in figure 2) can, on the other hand, be written in the form

$$\left[\partial_t + 2M - \frac{1}{p_0}\nabla^2_{\boldsymbol{r}}\right] C^{(1)}(\boldsymbol{r},t) = -g_\mathrm{E}^2 C_F p_0^{-2\epsilon} \mathcal{K}\left(\frac{1}{tp_0}, \frac{\nabla_{\boldsymbol{r}}}{p_0}, \frac{m_\mathrm{E}^2}{p_0^2}, \frac{\lambda^2}{p_0^2}, \boldsymbol{r}p_0\right) C^{(0)}(\boldsymbol{r},t) , \tag{6.6}$$

where the kernel $\mathcal{K}$ is dimensionless. Since the kernel is multiplied by g_E^2, we may assume that it can be expanded with respect to its first two arguments, given that non-trivial terms in this expansion should be subleading in g_E^2/p_0 with respect to the terms $\partial_t, -\nabla^2_{\boldsymbol{r}}/p_0$ already appearing in eq. (6.5). The zeroth order term turns out to involve a momentum space delta-function $\delta(q_3)$, which restricts the remaining loop integral to be $(2-2\epsilon)$-dimensional, and produces just the 1-loop static potential,

$$\begin{aligned} V(\boldsymbol{r}) &\equiv g_\mathrm{E}^2 C_F p_0^{-2\epsilon} \mathcal{K}\left(0, 0, \frac{m_\mathrm{E}^2}{p_0^2}, \frac{\lambda^2}{p_0^2}, \boldsymbol{r}p_0\right) \\ &= g_\mathrm{E}^2 C_F \int \frac{\mathrm{d}^{2-2\epsilon}q}{(2\pi)^{2-2\epsilon}} \left\{ \frac{1}{q^2+\lambda^2}\left[1 - e^{i\boldsymbol{q}\cdot\boldsymbol{r}}\right] - \frac{1}{q^2+m_\mathrm{E}^2}\left[1 + e^{i\boldsymbol{q}\cdot\boldsymbol{r}}\right] \right\} . \end{aligned} \tag{6.7}$$

Writing now $C(\boldsymbol{r},t) = C^{(0)}(\boldsymbol{r},t) + C^{(1)}(\boldsymbol{r},t)$, we can combine eqs. (6.5), (6.6), (6.7) to

$$\left[\partial_t + 2M - \frac{1}{p_0}\nabla^2_{\boldsymbol{r}} + V(\boldsymbol{r})\right] C(\boldsymbol{r},t) \propto \delta(t)\delta^{(2-2\epsilon)}(\boldsymbol{r}) , \tag{6.8}$$

where the point-splitting can be removed, $\boldsymbol{r} \to 0$, after the solution, and subleading terms have been dropped. Therefore, the exponential falloff is determined by the eigenvalues of the hamiltonian read from eq. (6.8).

It is worthwhile to note, in passing, the signs appearing in the two terms in eq. (6.7) (the first from A_3, the second from A_0). The reason for the differences are the ways in which A_3, A_0 appear in eq. (4.25): $\pm i g_{\rm E} A_3$ versus $-g_{\rm E} A_0$. The fact that the $\boldsymbol{r}$-independent terms from A_3, A_0 come with opposite signs, was already met in eqs. (5.3), (5.7).

Employing the basic integrals

$$\int \frac{{\rm d}^{2-2\epsilon} q}{(2\pi)^{2-2\epsilon}} \frac{1}{q^2+\lambda^2} = \frac{\mu^{-2\epsilon}}{4\pi}\left(\frac{1}{\epsilon} + 2\ln\frac{\bar\mu}{\lambda} + \mathcal{O}(\epsilon)\right), \tag{6.9}$$

$$\int \frac{{\rm d}^2 q}{(2\pi)^2} \frac{e^{i\boldsymbol{q}\cdot\boldsymbol{r}}}{q^2+\lambda^2} = \frac{1}{2\pi} K_0(\lambda r), \tag{6.10}$$

where $\bar\mu$ is the $\overline{\rm MS}$ scale parameter, $\mu = \bar\mu (e^\gamma/4\pi)^{1/2}$, and K_0 is a modified Bessel function, as well as the asymptotic behaviour of K_0,

$$K_0(x) = -\ln\frac{x}{2} - \gamma_E + \mathcal{O}(x), \tag{6.11}$$

it is observed that eq. (6.7) is both ultraviolet finite ($\lim_{\epsilon\to 0}$ exists) and infrared finite ($\lim_{\lambda\to 0}$ exists). Its final form is

$$V(\boldsymbol{r}) = g_{\rm E}^2 \frac{C_F}{2\pi}\left[\ln\frac{m_{\rm E} r}{2} + \gamma_E - K_0(m_{\rm E} r)\right]. \tag{6.12}$$

The potential determines the coefficient of the exponential fall-off, $\xi^{-1} \equiv m_{\rm full}$, through

$$\left[2M - \frac{\nabla_{\boldsymbol{r}}^2}{p_0} + V(\boldsymbol{r})\right]\Psi_0 = m_{\rm full}\Psi_0, \tag{6.13}$$

where Ψ_0 is the ground state wave function. Therefore, $m_{\rm full}$ is indeed insensitive to the gluonic ultrasoft scale at this order: there are no ambiguities inside the logarithm in eq. (4.26).

In order to determine $m_{\rm full}$, we have to solve eq. (6.13). We are not aware of an analytic solution for this bound state problem. In order to find the solution numerically, we rescale

$$r \equiv \frac{\hat r}{m_{\rm E}}, \quad m_{\rm full} - 2M \equiv g_{\rm E}^2 \frac{C_F}{2\pi}\hat E_0, \tag{6.14}$$

whereby eq. (6.13) becomes

$$\left[-\left(\frac{{\rm d}^2}{{\rm d}\hat r^2} + \frac{1}{\hat r}\frac{\rm d}{{\rm d}\hat r}\right) + \rho\left(\ln\frac{\hat r}{2} + \gamma_E - K_0(\hat r) - \hat E_0\right)\right]\Psi_0 = 0, \tag{6.15}$$

where

$$\rho = \frac{p_0 g_{\rm E}^2 C_F}{2\pi m_{\rm E}^2} = \frac{3(N_c^2-1)}{2N_c(2N_c+N_f)} = \begin{cases} 2/3, & N_f = 0 \\ 1/2, & N_f = 2 \\ 4/9, & N_f = 3. \end{cases} \tag{6.16}$$

A numerical solution of eq. (6.15) is facilitated by determining the analytic behaviour of $\Psi_0(\hat r)$ around the origin. Assuming $\Psi_0(0)$ to be finite, one finds

$$\Psi_0(\hat r) \approx \Psi_0(0)\left[1 + \frac{1}{2}\rho\,\hat r^2\left(\ln\frac{\hat r}{2} + \gamma_E - 1 - \frac{1}{2}\hat E_0\right)\right]. \tag{6.17}$$

Integrating towards large $\hat{r}$ and requiring square integrability, gives then

$$\begin{aligned} \hat{E}_0 &= 0.16368014, \quad \rho = 2/3\,, \quad (N_f = 0) &(6.18)\\ &= 0.38237416, \quad \rho = 1/2\,, \quad (N_f = 2) &(6.19)\\ &= 0.46939139, \quad \rho = 4/9\,. \quad (N_f = 3) &(6.20)\end{aligned}$$

Therefore, for $N_c = 3$, $N_f = 0$ (corresponding to quenched QCD), eqs. (5.9), (6.14) and (6.18) together imply that

$$m_{\rm full} = 2\pi T + g_{\rm E}^2 \frac{C_F}{2\pi}\left(\frac{1}{2} + \hat{E}_0\right) \approx 2\pi T + 0.14083730\, g^2 T\,. \tag{6.21}$$

It is worthwhile to note that this expression contains no logarithms. Here we differ from [26], for example, where the contributions from A_0 were left out. As can be seen in eqs. (6.7), (6.12), these contributions are essential.

At $T \sim 2T_c$ for $N_f = 0$, it is estimated [5] that the effective gauge coupling is $g_{\rm E}^2/T \approx 2.7$. Thus the second term in eq. (6.21) represents a $\sim 5\%$ correction to the leading order result, with a positive sign. Since the spinor structure of the operators in eqs. (4.10)–(4.17) played no significant role in the analysis above (it only leads to an overall factor, $\mathrm{Tr}\,1$ or $\mathrm{Tr}\,\sigma_3^2$), the same result holds for all the flavour non-singlet operators of the type $\phi^\dagger\chi$, that is, S^n, P^n, V_k^n, A_k^n. As we have mentioned, the correlators related to V_0^n, A_0^n are suppressed, while those related to V_3^n, A_3^n are constants because of current conservation, and vanish in infinite volume, such that no meaningful long-distance screening masses can be associated with them.

7. Comparison with lattice

Let us now consider lattice measurements of the correlation lengths discussed, at $T > T_c$. This is a classic problem in lattice QCD, initiated already a long time ago [25]. Most measurements concentrate on the non-singlets P^n, V_μ^n (or "π, ρ"), because the singlets require an additional "disconnected" contraction (corresponding to the quark lines in figure 2(c)), whose inclusion is non-trivial. For recent original work on the topic, we refer to [40]–[42], and for a recent review, to [16].

As always with lattice simulations, practical measurements are faced with finite volume and discretisation artifacts. The measurements we are concerned with face also further problems, since the operators are fermionic: we would like to approach the chiral limit and simultaneously respect the flavour symmetries, which turn out to be hard requirements to satisfy. Simulations could be carried out with, say, staggered, Wilson, domain wall, or overlap quarks, but in all cases but the last, which is very expensive in practice, it is oftentimes difficult to estimate how serious the systematic errors are. Finally, most lattice simulations are "quenched", ignoring the fermion determinant, which we however do not need to consider a restriction, since we can easily set $N_f = 0$ in our predictions.

To summarise the current status, from quenched measurements with Wilson quarks at $N_\tau = 8$ [40]: already at $T \sim 1.5T_c$, the π is just slightly below ρ. Both have masses consistent with or very slightly below the free theory prediction, that is $2\pi T$ in the continuum.

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Very close to T_c the situation has to be different, of course, and the pion mass should even vanish at T_c while approaching the chiral limit with dynamical quarks, for $N_f = 2$.

This pattern is quite different from what it used to be: in the early days, staggered quarks gave an anomalously low screening mass for the pion even at $T > T_c$, both in the quenched as well as in the unquenched case. It has been found out, however, that discretisation effects are substantial for staggered quarks: in the quenched theory, the light masses increase in going from $N_\tau = 4$ to $N_\tau = 12$, at $T = (1.5 \ldots 2.0)T_c$, by up to $\sim$ 40% (when masses are expressed in units of T) [41, 23]. On the other hand, the effects of quenching seem to be moderate, although only tested far from the continuum limit (for $N_\tau = 4$, staggered quarks) [42].

To summarise, within the current resolution the mesonic screening masses agree with the leading order prediction, within statistical and certainly within systematic errors. It will require a significantly higher resolution to show unambiguously whether the 5% correction we have found is really reproduced by the lattice data in the chiral continuum limit. In any case, the classic problem, that the π and ρ seemed not to be degenerate at $T > T_c$ in contrast to the perturbative prediction, has disappeared while approaching the continuum limit.

8. Flavour singlets

Flavour singlet quark bilinears behave in a different way than flavour non-singlets, because their correlation functions contain a "disconnected" contribution, of the type in figure 2c. This means that the operators couple, in general, to bosonic glueball states. They are also more difficult to measure on the lattice, just because of the disconnected contraction. Here we discuss some basic features concerning the flavour singlet correlators.

The relations of the flavour singlets to gluonic operators can be determined by coupling the flavour singlets to sources in the lagrangian, and then performing the Grassmann integral over the quarks, in a given gauge field background. At high temperatures, this procedure is infrared safe, and the results can be expanded in $m^2/(\pi T)^2$. To leading order in the expansion, a 1-loop computation of 0-gluon, 1-gluon, 2-gluon and 3-gluon vertices leads to

$$\frac{1}{N_f}\bar\psi\psi \leftrightarrow N_c\frac{mT^2}{6} - \frac{m}{2\pi^2}g^2 \operatorname{Tr} A_0^2\,, \tag{8.1}$$

$$\frac{1}{N_f}\bar\psi\gamma_5\psi \leftrightarrow \frac{7\zeta(3)m}{8\pi^2T^2}\frac{g^2}{32\pi^2}\epsilon_{\alpha\beta\gamma\delta}F^a_{\alpha\beta}F^a_{\gamma\delta}\,, \tag{8.2}$$

$$\frac{1}{N_f}\bar\psi\gamma_0\psi \leftrightarrow -\frac{i}{3\pi^2}g^3 \operatorname{Tr} A_0^3\,, \tag{8.3}$$

$$\frac{1}{N_f}\bar\psi\gamma_i\psi \leftrightarrow 0\,. \tag{8.4}$$

These relations imply, for instance, that the correlation length of $S^s = \bar\psi\psi$ can be determined from $\operatorname{Tr} A_0^2$, and the correlation length of the baryon density $V_0^s = \bar\psi\gamma_0\psi$ from

$\mathrm{Tr}\, A_0^3$ [9, 44]. Furthermore, from

$$\frac{g^2}{32\pi^2}\epsilon_{\alpha\beta\gamma\delta}F^a_{\alpha\beta}F^a_{\gamma\delta} = \partial_\mu K_\mu \,, \tag{8.5}$$

$$K_\mu = \frac{g^2}{8\pi^2}\epsilon_{\mu\nu\lambda\rho}\left[A^a_\nu\partial_\lambda A^a_\rho + \frac{g}{3}f^{abc}A^a_\nu A^b_\lambda A^c_\rho\right], \tag{8.6}$$

it follows by integrating over $\int \mathrm{d}\tau\, \mathrm{d}^2\boldsymbol{x}_\perp$ that the correlation length of $P^s = \bar\psi\gamma_5\psi$ is determined by that of $K_3 \propto g^2\epsilon_{ij3}A^a_0F^a_{ij}$, where the form indicated applies in the static limit. All of these gluonic correlation lengths have been measured, for various N_f, in ref. [9]. It may also be noted from eqs. (8.1), (8.2) that some of the couplings vanish in the chiral limit.

For the axial charge $A^s_0 = \bar\psi\gamma_0\gamma_5\psi$, the relation to gluonic objects is very non-trivial because of the axial anomaly. We recall here only that if one couples the axial charge to a constant chemical potential, and then integrates over spacetime, one obtains [43]

$$\frac{1}{N_f}\bar\psi\gamma_0\gamma_5\psi \leftrightarrow K_0 \,. \tag{8.7}$$

As is well-known this relation does not unambiguously apply for local operators, however (the right-hand-side, the Chern-Simons number density, is not even gauge-invariant). Nevertheless it shows explicitly the quantum numbers of the gluonic objects that A^s_0 may couple to.

Let us finally discuss the spatial components of the vector and axial currents, and also touch the issue of $\mathrm{U_A}(1)$ axial symmetry non-restoration (for a review see, e.g., [47]). To start with, we may recall that $\bar\psi\gamma_\mu\gamma_5\psi$ and $\bar\psi\gamma_5\psi$ are related through the anomaly equation

$$\partial_\mu[\bar\psi\gamma_\mu\gamma_5\psi] = 2m\,\bar\psi\gamma_5\psi + N_f\frac{g^2}{32\pi^2}\epsilon_{\alpha\beta\gamma\delta}F^a_{\alpha\beta}F^a_{\gamma\delta} \,. \tag{8.8}$$

Indeed, this relation remains true even at finite temperatures [45], in spite of the fact that the ways in which $\partial_\mu[\bar\psi\gamma_\mu\gamma_5\psi]$ and $\bar\psi\gamma_5\psi$ separately couple to gluonic operators, change [46]. Integrating now (the chiral $m\to 0$ limit of) eq. (8.8) over $\int \mathrm{d}\tau \mathrm{d}^2\boldsymbol{x}_\perp$ and noticing that the partial derivatives ∂_3 appearing on both sides do not modify the qualitative structure and can thus be dropped out, it is observed that the correlators in the x_3-direction of $A^s_3 = \bar\psi\gamma_3\gamma_5\psi$ are determined by the operator $K_3 \propto g^2\epsilon_{ij3}A^a_0F^a_{ij}$. Thus the correlator of A^s_3 falls off exponentially, with a correlation length determined [48, 6] to be $\xi^{-1}[A^s_3] \equiv m[A^s_3] = m[K_3] \approx m_\mathrm{E} + g^2_\mathrm{E}N_c(\ln(m_\mathrm{E}/g^2_\mathrm{E}) + 7.0)/4\pi + \mathcal{O}(g^3T)$. On the contrary, the vector current is still conserved, and thus the correlator related to $V^s_3 = \bar\psi\gamma_3\psi$ is a constant, and vanishes in infinite volume. Thus, the long distance screening masses related to the $\mathrm{U_A}(1)$ partners A^s_3, V^s_3 are different even at arbitrarily high temperatures and, in this sense, the axial $\mathrm{U_A}(1)$ symmetry is strictly speaking never restored. A similar conclusion of course also holds, for instance, for S^s, P^s, as discussed above.

9. Conclusions

The purpose of this paper has been to discuss analytic predictions for the screening masses related to various quark bilinears at high temperatures. We have addressed both flavour singlets, for which the screening masses are determined by those measured with bosonic glueball operators in the dimensionally reduced theory, and flavour non-singlets, for which we have determined the next-to-leading-order perturbative correction, eq. (6.21).

The next-to-leading-order correction we find is small, but positive in sign. This is in contrast to what lattice measurements traditionally indicated, and leads to a non-monotonic behaviour for the mesonic screening masses, whereby they first "overshoot" the perturbative result $2\pi T$ as the temperature is increased above T_c, and then approach it from above for $T \gg T_c$. It will be interesting to see whether this kind of a behaviour can be reproduced, as the lattice studies approach the infinite volume, continuum, and chiral limits. We should stress that our predictions hold formally also for the quenched theory, probably the most immediate case on which progress could be expected.

Apart from 4d lattice simulations, another interesting direction for further study might be to implement the effective theory we have derived, NRQCD$_3$, in lattice regularisation, and measure the correlation lengths that way, following [49] (assuming that the non-trivial issues of renormalization and numerical instabilities in NRQCD$_3$ can be dealt with). This would allow to account for some higher order corrections as well as non-perturbative effects, showing in particular whether those could change the sign or the magnitude of the correction we have discussed, at realistic temperatures not far above the deconfinement phase transition, where the coupling constant is not asymptotically small.

The ultimate theoretical goal of these studies is to understand the dynamics of QCD at temperatures above a few hundred MeV, relevant for cosmology as well as for heavy ion collision experiments. One may ask, in particular, whether the strict coupling constant expansion might work for some observables after all, when carried out to a sufficient depth (possibly including non-perturbative coefficients), in spite of the fact that the first few corrections are huge [50, 51, 11, 5]. This would allow to avoid the use of more complicated and partly phenomenological tools (for which many alternatives are available, however [18]). Apart from gluonic observables [12]–[17], there are promising signs for this also from related fermionic observables, quark number susceptibilities [24, 13], but our case is more infrared sensitive and in this sense a stronger test.

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References

[1] A.D. Linde, *Infrared problem in thermodynamics of the Yang-Mills gas*, *Phys. Lett.* **B 96** (1980) 289.

[2] D.J. Gross, R.D. Pisarski and L.G. Yaffe, *QCD and instantons at finite temperature*, *Rev. Mod. Phys.* **53** (1981) 43.

[3] P. Ginsparg, *First order and second order phase transitions in gauge theories at finite temperature*, *Nucl. Phys.* **B 170** (1980) 388;

T. Appelquist and R.D. Pisarski, *High-temperature Yang-Mills theories and three-dimensional quantum chromodynamics*, *Phys. Rev.* **D 23** (1981) 2305.

[4] S. Nadkarni, *Large scale structure of the deconfined phase*, *Phys. Rev. Lett.* **60** (1988) 491;
T. Reisz, *Realization of dimensional reduction at high temperature*, *Z. Physik* **C 53** (1992) 169;
L. Kärkkäinen, P. Lacock, D.E. Miller, B. Petersson and T. Reisz, *Dimensional reduction in* SU(3) *gauge theory*, *Phys. Lett.* **B 282** (1992) 121; *The physical phase of dimensionally reduced gauge theories*, *Nucl. Phys.* **B 418** (1994) 3 [hep-lat/9310014];
L. Kärkkäinen, P. Lacock, B. Petersson and T. Reisz, *Dimensional reduction and color screening in QCD*, *Nucl. Phys.* **B 395** (1993) 733.

[5] K. Kajantie, M. Laine, K. Rummukainen and M.E. Shaposhnikov, *3d* $\mathrm{SU}(N)$ *+ adjoint Higgs theory and finite-temperature QCD*, *Nucl. Phys.* **B 503** (1997) 357 [hep-ph/9704416].

[6] M. Laine and O. Philipsen, *Gauge-invariant scalar and field strength correlators in 3d*, *Nucl. Phys.* **B 523** (1998) 267 [hep-lat/9711022]; *The non-perturbative QCD Debye mass from a Wilson line operator*, *Phys. Lett.* **B 459** (1999) 259 [hep-lat/9905004].

[7] A. Hart and O. Philipsen, *The spectrum of the three-dimensional adjoint Higgs model and hot* SU(2) *gauge theory*, *Nucl. Phys.* **B 572** (2000) 243 [hep-lat/9908041].

[8] P. Bialas, A. Morel, B. Petersson, K. Petrov and T. Reisz, *High temperature 3d QCD: dimensional reduction at work*, *Nucl. Phys.* **B 581** (2000) 477 [hep-lat/0003004].

[9] A. Hart, M. Laine and O. Philipsen, *Static correlation lengths in QCD at high temperatures and finite densities*, *Nucl. Phys.* **B 586** (2000) 443 [hep-ph/0004060]; *Testing imaginary vs. real chemical potential in finite-temperature QCD*, *Phys. Lett.* **B 505** (2001) 141 [hep-lat/0010008].

[10] A. Cucchieri, F. Karsch and P. Petreczky, *Propagators and dimensional reduction of hot* SU(2) *gauge theory*, *Phys. Rev.* **D 64** (2001) 036001 [hep-lat/0103009].

[11] E. Braaten and A. Nieto, *Free energy of QCD at high temperature*, *Phys. Rev.* **D 53** (1996) 3421 [hep-ph/9510408].

[12] K. Kajantie, M. Laine, K. Rummukainen and Y. Schröder, *How to resum long-distance contributions to the QCD pressure?*, *Phys. Rev. Lett.* **86** (2001) 10 [hep-ph/0007109]; *Four-loop logarithms in 3d gauge + Higgs theory*, *Nucl. Phys.* **119** *(Proc. Suppl.)* (2003) 577 [hep-lat/0209072]; *The pressure of hot QCD up to* $g^6 \ln(1/g)$, *Phys. Rev.* **D 67** (2003) 105008 [hep-ph/0211321]; *Four-loop vacuum energy density of the* $\mathrm{SU}(N_c)$ *+ adjoint Higgs theory*, *J. High Energy Phys.* **04** (2003) 036 [hep-ph/0304048];
K. Kajantie, M. Laine and Y. Schröder, *A simple way to generate high order vacuum graphs*, *Phys. Rev.* **D 65** (2002) 045008 [hep-ph/0109100];
Y. Schröder, *Automatic reduction of four-loop bubbles*, *Nucl. Phys.* **116** *(Proc. Suppl.)* (2003) 402 [hep-ph/0211288]; *Tackling the infrared problem of thermal QCD*, hep-lat/0309112;
F. Di Renzo, A. Mantovi, V. Miccio and Y. Schröder, *Four loop stochastic perturbation theory in 3d* SU(3), hep-lat/0309111;
Y. Schröder and A. Vuorinen, *High-precision evaluation of four-loop vacuum bubbles in three dimensions*, hep-ph/0311323.

[13] A. Vuorinen, *The pressure of QCD at finite temperatures and chemical potentials*, *Phys. Rev.* **D 68** (2003) 054017 [hep-ph/0305183].

JHEP02(2004)004

[14] J.P. Blaizot, E. Iancu and A. Rebhan, *On the apparent convergence of perturbative QCD at high temperature*, *Phys. Rev.* **D 68** (2003) 025011 [hep-ph/0303045];
A. Ipp, A. Rebhan and A. Vuorinen, *Perturbative QCD at non-zero chemical potential: comparison with the large-N_f limit and apparent convergence*, hep-ph/0311200.

[15] M. Laine, *What is the simplest effective approach to hot QCD thermodynamics?*, hep-ph/0301011.

[16] E. Laermann and O. Philipsen, *Status of lattice QCD at finite temperature*, hep-ph/0303042.

[17] C.P. Korthals Altes, *Forces, fluxes and quasi-particles in hot QCD*, hep-ph/0308229.

[18] M. Strickland, *Reorganizing finite temperature field theory*, *Int. J. Mod. Phys.* **A 16S1C** (2001) 1277;
E. Braaten, *Thermodynamics of hot QCD*, *Nucl. Phys.* **A 702** (2002) 13;
A. Peshier, *Resummation of the QCD thermodynamic potential*, *Nucl. Phys.* **A 702** (2002) 128 [hep-ph/0110342];
J.O. Andersen, *Hard thermal loops and QCD thermodynamics*, hep-ph/0210195;
J.-P. Blaizot, E. Iancu and A. Rebhan, *Thermodynamics of the high-temperature quark gluon plasma*, hep-ph/0303185;
U. Kraemmer and A. Rebhan, *Advances in perturbative thermal field theory*, hep-ph/0310337.

[19] S. Gottlieb, W. Liu, D. Toussaint, R.L. Renken and R.L. Sugar, *The quark number susceptibility of high temperature QCD*, *Phys. Rev. Lett.* **59** (1987) 2247.

[20] J.P. Blaizot, E. Iancu and A. Rebhan, *Quark number susceptibilities from HTL-resummed thermodynamics*, *Phys. Lett.* **B 523** (2001) 143 [hep-ph/0110369]; *Comparing different hard-thermal-loop approaches to quark number susceptibilities*, *Eur. Phys. J.* **C 27** (2003) 433 [hep-ph/0206280].

[21] P. Chakraborty, M.G. Mustafa and M.H. Thoma, *Quark number susceptibility in hard thermal loop approximation*, *Eur. Phys. J.* **C 23** (2002) 591 [hep-ph/0111022]; *Chiral susceptibility in hard thermal loop approximation*, *Phys. Rev.* **D 67** (2003) 114004 [hep-ph/0210159].

[22] R.V. Gavai and S. Gupta, *The continuum limit of quark number susceptibilities*, *Phys. Rev.* **D 65** (2002) 094515 [hep-lat/0202006];
MILC collaboration, C. Bernard et al., *High temperature QCD with three flavors of improved staggered quarks*, *Nucl. Phys.* **119** *(Proc. Suppl.)* (2003) 523 [hep-lat/0209079].

[23] R.V. Gavai and S. Gupta, *Valence quarks in the QCD plasma: quark number susceptibilities and screening*, *Phys. Rev.* **D 67** (2003) 034501 [hep-lat/0211015]; *Pressure and non-linear susceptibilities in QCD at finite chemical potentials*, *Phys. Rev.* **D 68** (2003) 034506 [hep-lat/0303013].

[24] A. Vuorinen, *Quark number susceptibilities of hot QCD up to $g^6 \ln(g)$*, *Phys. Rev.* **D 67** (2003) 074032 [hep-ph/0212283].

[25] C. DeTar and J.B. Kogut, *The hadronic spectrum of the quark plasma*, *Phys. Rev. Lett.* **59** (1987) 399;
C. DeTar and J.B. Kogut, *Measuring the hadronic spectrum of the quark plasma*, *Phys. Rev.* **D 36** (1987) 2828.

[26] T.H. Hansson and I. Zahed, *Hadronic correlators in hot QCD*, *Nucl. Phys.* **B 374** (1992) 277;
T.H. Hansson, M. Sporre and I. Zahed, *Baryonic and gluonic correlators in hot QCD*, *Nucl. Phys.* **B 427** (1994) 545 [hep-ph/9401281].

[27] V. Koch, E.V. Shuryak, G.E. Brown and A.D. Jackson, *The propagation of quarks in the spatial direction in hot QCD*, *Phys. Rev.* **D 46** (1992) 3169 [hep-ph/9204236];
V. Koch, *On the temperature dependence of correlation functions in the space-like direction in hot QCD*, *Phys. Rev.* **D 49** (1994) 6063 [hep-ph/9401284].

[28] M. Ishii and T. Hatsuda, *Hadronic screening masses and the magnetic gluon condensate at high temperature*, *Phys. Lett.* **B 338** (1994) 319 [hep-ph/9408209].

[29] J. Gasser and H. Leutwyler, *Light quarks at low temperatures*, *Phys. Lett.* **B 184** (1987) 83.

[30] V.L. Eletsky and B.L. Ioffe, *On temperature dependence of correlators of hadronic currents*, *Sov. J. Nucl. Phys.* **48** (1988) 384.

[31] S.-Z. Huang and M. Lissia, *The dimensionally reduced effective theory for quarks in high temperature QCD*, *Nucl. Phys.* **B 480** (1996) 623 [hep-ph/9511383].

[32] H.-G. Dosch, J. Kripfganz, A. Laser and M.G. Schmidt, *Bound states in the hot electroweak phase*, *Phys. Lett.* **B 365** (1996) 213 [hep-ph/9509352]; *Non-perturbative correlation masses in the hot electroweak phase*, *Nucl. Phys.* **B 507** (1997) 519 [hep-ph/9612450].

[33] W.E. Caswell and G.P. Lepage, *Effective lagrangians for bound state problems in QED, QCD and other field theories*, *Phys. Lett.* **B 167** (1986) 437.

[34] S.-Z. Huang and M. Lissia, *The relevant scale parameter in the high temperature phase of QCD*, *Nucl. Phys.* **B 438** (1995) 54 [hep-ph/9411293].

[35] K. Kajantie, M. Laine, K. Rummukainen and M.E. Shaposhnikov, *Generic rules for high temperature dimensional reduction and their application to the Standard Model*, *Nucl. Phys.* **B 458** (1996) 90 [hep-ph/9508379].

[36] M. Neubert, *Heavy-quark effective theory*, hep-ph/9610385.

[37] A.V. Manohar, *The HQET/NRQCD lagrangian to order α/m^3*, *Phys. Rev.* **D 56** (1997) 230 [hep-ph/9701294].

[38] A. Pineda and J. Soto, *Effective field theory for ultrasoft momenta in NRQCD and NRQED*, *Nucl. Phys.* **64** *(Proc. Suppl.)* (1998) 428 [hep-ph/9707481]; *Potential NRQED: the positronium case*, *Phys. Rev.* **D 59** (1999) 016005 [hep-ph/9805424];
N. Brambilla, A. Pineda, J. Soto and A. Vairo, *Potential NRQCD: an effective theory for heavy quarkonium*, *Nucl. Phys.* **B 566** (2000) 275 [hep-ph/9907240].

[39] M. Beneke and V.A. Smirnov, *Asymptotic expansion of Feynman integrals near threshold*, *Nucl. Phys.* **B 522** (1998) 321 [hep-ph/9711391];
M. Beneke, A. Signer and V.A. Smirnov, *Top quark production near threshold and the top quark mass*, *Phys. Lett.* **B 454** (1999) 137 [hep-ph/9903260];
M. Beneke, *Perturbative heavy quark-antiquark systems*, hep-ph/9911490.

[40] P.M. Vranas, *The finite temperature QCD phase transition with domain wall fermions*, *Nucl. Phys.* **83** *(Proc. Suppl.)* (2000) 414 [hep-lat/9911002];
QCD-TARO collaboration, P. de Forcrand et al., *Meson correlators in finite temperature lattice QCD*, *Phys. Rev.* **D 63** (2001) 054501 [hep-lat/0008005];
E. Laermann and P. Schmidt, *Meson screening masses at high temperature in quenched QCD with improved Wilson quarks*, *Eur. Phys. J.* **C 20** (2001) 541 [hep-lat/0103037].

[41] R.V. Gavai, S. Gupta and R. Lacaze, *Quenched QCD at finite temperature with chiral fermions*, *Phys. Rev.* **D 65** (2002) 094504 [hep-lat/0107022];
K. Nomura, O. Miyamura, T. Umeda and H. Matsufuru, *Study of spatial meson correlators at finite temperature in quenched anisotropic lattice QCD*, *Nucl. Phys.* **106** *(Proc. Suppl.)* (2002) 507 [hep-lat/0110204].

[42] R.V. Gavai, S. Gupta and P. Majumdar, *Susceptibilities and screening masses in two flavor QCD*, *Phys. Rev.* **D 65** (2002) 054506 [hep-lat/0110032].

[43] A.N. Redlich and L.C.R. Wijewardhana, *Induced Chern-Simons terms at high temperatures and finite densities*, *Phys. Rev. Lett.* **54** (1985) 970.

[44] D. Bödeker and M. Laine, *Finite baryon density effects on gauge field dynamics*, *J. High Energy Phys.* **09** (2001) 029 [hep-ph/0108034].

[45] H. Itoyama and A.H. Mueller, *The axial anomaly at finite temperature*, *Nucl. Phys.* **B 218** (1983) 349.

[46] C. Contreras and M. Loewe, *The triangle diagram at finite temperature*, *Z. Physik* **C 40** (1988) 253.

[47] M. Marchi and E. Meggiolaro, *Reviewing the problem of the* U(1) *axial symmetry and the chiral transition in QCD*, *Nucl. Phys.* **B 665** (2003) 425 [hep-ph/0301247].

[48] P. Arnold and L.G. Yaffe, *The nonabelian Debye screening length beyond leading order*, *Phys. Rev.* **D 52** (1995) 7208 [hep-ph/9508280].

[49] B.A. Thacker and G.P. Lepage, *Heavy quark bound states in lattice QCD*, *Phys. Rev.* **D 43** (1991) 196;
C.T.H. Davies and B.A. Thacker, *Heavy quark renormalization parameters in nonrelativistic QCD*, *Phys. Rev.* **D 45** (1992) 915;
G.P. Lepage, L. Magnea, C. Nakhleh, U. Magnea and K. Hornbostel, *Improved nonrelativistic QCD for heavy quark physics*, *Phys. Rev.* **D 46** (1992) 4052 [hep-lat/9205007].

[50] P. Arnold and C.-X. Zhai, *The three loop free energy for pure gauge QCD*, *Phys. Rev.* **D 50** (1994) 7603 [hep-ph/9408276]; *The three loop free energy for high temperature QED and QCD with fermions*, *Phys. Rev.* **D 51** (1995) 1906 [hep-ph/9410360].

[51] C. Zhai and B. Kastening, *The free energy of hot gauge theories with fermions through* g^5, *Phys. Rev.* **D 52** (1995) 7232 [hep-ph/9507380].